AN EXPLICIT EXAMPLE OF OPTIMAL PORTFOLIO-CONSUMPTION CHOICES WITH HABIT FORMATION AND PARTIAL OBSERVATIONS

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Abstract. We consider a model of optimal investment and consumption with both habit formation and partial observations in incomplete Itô processes markets. The individual investor develops addictive consumption habits gradually while only observing the market stock prices but not the instantaneous rate of return, which follows an Ornstein-Uhlenbeck process. Applying the Kalman-Bucy filtering theorem and Dynamic Programming arguments, we solve the associated HJB equation explicitly for this path dependent stochastic control problem in the case of power utility preferences. We provide the optimal investment and consumption policies in explicit feedback forms using rigorous verification arguments.

1. Introduction

Habit formation has become a popular alternative tool for modeling preferences on optimal consumption streams during recent years. It has been observed that the time separable property of traditional von Neumann-Morgenstern utility is not consistent with many empirical experiments, for instance, the celebrated Equity Premium Puzzle. (see *Mehra and Prescott* [15] and *Constantinides* [4]). The financial economics literature has been arguing that the past consumption pattern has a continuing impact on individual's current consumption decisions, and the preference should depend on both the consumption rate process and the corresponding consumption history. In particular, the consumption path dependent linear habit formation preference $\mathbb{E}[\int_0^T U(t, c_t - Z_t)dt]$ has been widely accepted, where the index Z_t stands for the accumulative consumption history. The definition that instantaneous utility function is decreasing in Z_t indicates that an increase in consumption today increases current utility but depresses all future utilities through the induced increase in future standards of living.

The continuous time utility maximization problem with linear *addictive habit* formation in the complete Itô processes market has been extensively studied in the past decades, see for instance, *Detemple and Zapatero* [5] and [6], *Schroder and Skiadas* [20], *Munk* [17] and *Englezos and Karatzas* [7]. Recently, this continuous time path dependent optimization problem has been solved in the general incomplete semimartingale market by *Yu* [23], using the path-dependence reduction transform and convex duality analysis.

The contributions of the present work are two-fold. *1*). From the modeling perspective, we are considering the habits forming investor in incomplete Itô processes financial markets, together

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with additional partial observations constraint. We are facing the case that the individual investor develops his own consumption habits during the whole investment period and meanwhile has only access to the public stock price information. In other words, he can not observe the mean rate of return process μ_t and the corresponding Brownian motion W_t which appears in the stock price dynamics. In our model, we will assume μ_t follows the mean reverting Ornstein Uhlenbeck process driven by another Brownian motion B_t which is not perfectly correlated with W_t. Subject to the loss of information and filtration shrinkage, this stochastic control problem can not be covered by the main theorems in *Yu* [23], which motivates the separate work of this paper. *2*). On the other hand, at the mathematical level, we solve the relatively complicated nonlinear HJB equation fully explicitly. As a consequence, we furthermore derive the $\mathcal{F}_t^S$-adapted optimal investment and consumption polices in feedback form via rigorous verification arguments. Our analytical approach allows us to avoid proving the Dynamic Programming Principle and the measurable selection arguments associated with it.

Optimal investment problems under incomplete information have been studied by numerous authors, and we only list a very small subset of them: *Lakner* [14] applies martingale methods and derived the structure of the optimal investment strategies. The linear diffusion model is studied by *Brendle* [2], who derives explicit results for the value of information on optimal investment with power and exponential utilities. The effects of learning on the composition of the optimal portfolios are studied in *Brennan* [3] and *Xia* [22]. *Monoyios* [16] considers the optimal investment with both the uncertainty of the drift parameter and the inside information of the the Brownian motion at terminal time, for which he obtains an explicit solution via Kalman-Bucy filtering together with techniques of enlargement of the filtration. *Björk, Davis and Landén* [1] examine the market model with unobservable rates of returns that are arbitrary semimartingales, and they provide a unified treatment for a large class of partially observed investment problems.

To the best of our knowledge, the utility maximization with consumption habit formation under incomplete information is not yet addressed in the literature. However, we want to single out the work of *Munk* [17] and *Brendle* [2] which are technically close to our problem. *Munk* [17] tackles the utility maximization with consumption habit formation in the complete market, where he assumes the market price of risk process obeys a mean reverting SDE, moreover, the stock price process and the drift process are driven by the same Brownian motion. By applying the Market Isomorphism result by *Schroder and Skiadas* [20] to the work by *Wachter* [21], he obtains the explicit solution for the HJB equation and optimal control policies under power utility preference for the case $p < 0$. However, the complete market setting is too restrictive. On the other hand, *Brendle* [2] treats the problem of utility maximization on the terminal wealth in the incomplete Itô process market with partial observations. He figures out the solution of the HJB equation can be expressed in a closed form by solving some ODE systems with time dependent parameters. Moreover, by setting some technical substitutions, he proves that solving the previous ODEs is actually equivalent to solving a family of ODEs with constant parameters, whose solutions have been discussed earlier by *Kim and Omberg* [13] in a different problem setting. When the intertemporal consumption choice comes into play, together with path dependent habit formation impact, it is not clear whether the HJB equation can still admit an explicit solution. Our contribution can also be summarized as that

we combine the two models considered by the previous authors, and successfully solve the *Munk*'s habit formation problem in the setting of *Brendle*'s incomplete market with partial observations. Furthermore, we provide the rigorous verification of the main results which is missing in their previous work.

The structure of the present paper is outlined as: Section 2 introduces the market model and the concept of habit formation process.The utility maximization problem with addictive habit formation and partial observations is defined in Section 3. By applying the Kalman Bucy filtering theorem and Dynamic Programming arguments, we formally derive the Hamilton-Jacobi-Bellman(HJB) equation for the power utility preference and we provide the decoupled form solution of this nonlinear PDE, which reduces the algorithm to solving some auxiliary ODEs with constant coefficients. Based on these classical solutions, the explicit feedback form of the optimal investment and consumption policies will be obtained. Section 4 contains rigorous proofs of the corresponding verification arguments. At last, four cases of fully explicit solutions of some auxiliary ODEs are presented in the Appendix *A*.

2. Market Model and Consumption Habit Formation

On a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with the background filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$, which satisfies the usual conditions, we consider a financial market with one risk-free bond and one stock account for a "small investor" over a finite time horizon $[0, T]$. The price of the bond S_t^0 solves:

$$dS_t^0 = r_t S_t^0 dt, \quad 0 \leq t \leq T$$

with initial price $S_0^0 = 1$, and without loss of generality, we assume the interest rate $r_t \equiv 0$, for all $t \in [0, T]$, this can be achieved by the standard change of numéraire.

The stock price S_t is modeled as a diffusion process solving:

$$dS_t = \mu_t S_t dt + \sigma_S S_t dW_t, \quad 0 \leq t \leq T, \tag{2.1}$$

with $S_0 = s > 0$, where the drift process μ_t is $\mathbb{F}$ adapted, and satisfies the mean-reverting Ornstein Uhlenbeck SDE:

$$d\mu_t = -\lambda(\mu_t - \bar{\mu})dt + \sigma_\mu dB_t, \quad 0 \leq t \leq T. \tag{2.2}$$

Here, W_t and B_t are $\mathbb{F}$ adapted Brownian motions defined on $(\Omega, \mathbb{F}, \mathbb{P})$ and they are correlated with the coefficient $\rho \in [-1, 1]$. We assume the initial value of the drift process μ_0 is an $\mathcal{F}_0$ measurable Gaussian random variable, satisfying $\mu_0 \sim \mathrm{N}(\eta_0, \theta_0)$, which is independent of Brownian motions $(W_t)_{0\leq t\leq T}$ and $(B_t)_{0\leq t\leq T}$. We also assume all the coefficients $\sigma_S \geq 0, \lambda \geq 0, \bar{\mu}, \sigma_\mu \geq 0$ are constants.

Remark 2.1. *Under the full background filtration $\mathbb{F}$, we do not assume the existence of the equivalent local martingale measures in the market, for example, the exponential local martingale deflator defined by*

$$H_t = \exp\Big(-\int_0^t \frac{\mu_v}{\sigma_S} dW_v - \frac{1}{2}\int_0^t \frac{\mu_v^2}{\sigma_S^2} dv\Big), \quad 0 \leq t \leq T. \tag{2.3}$$

is allowed to be a strict local martingale.

Now, at each time $t \in [0,\ T]$, the investor chooses a consumption rate $c_t \geq 0$, and decides the amounts π_t of his wealth to invest in the stock, and the rest in bank. Then, in this self-financing market model, the investor's total wealth process X_t follows the dynamics:

$$dX_t = (\pi_t \mu_t - c_t)dt + \sigma_S \pi_t dW_t, \quad 0 \leq t \leq T, \tag{2.4}$$

with the initial wealth $X_0 = x_0 > 0$.

In this paper, we adopt the notation $Z_t = Z(t; c_t)$ as "Habit Formation" process or "the standard of living" process to describe his "consumption habits level" . We assume the accumulative index Z_t follows the dynamics:

$$dZ_t = (\delta(t)c_t - \alpha(t)Z_t)dt, \quad 0 \leq t \leq T, \tag{2.5}$$

where $Z_0 = z \geq 0$ is called the *initial habit.*

Equivalently, (2.5) stipulates

$$Z_t = ze^{-\int_0^t \alpha(u)du} + \int_0^t \delta(u)e^{-\int_u^t \alpha(v)dv} c_u du, \quad 0 \leq t \leq T, \tag{2.6}$$

and it is the exponentially weighted average of the initial habit and the past consumption. Here, the discounting factors $\alpha(t)$ and $\delta(t)$ measure, respectively, the persistence of the past level and the intensity of consumption history, and are assumed to be nonnegative continuous functions..

In our current work, we are only considering the case of "addictive habits", i.e., we require investor's current consumption strategies shall never fall below the standard of living level,

$$c_t \geq Z_t, \quad \forall 0 \leq t \leq T, \quad a.s.. \tag{2.7}$$

We will see in the future that this additional consumption constraint implies the initial wealth must be sufficiently large to sustain habits and ensure the existence of optimal policies.

3. Utility Maximization with Kalman-Bucy Filtering

3.1. **Dynamic Programming Arguments under Partial Observations Filtration $\mathbb{F}^S$.** From now on, we shall make the assumption that the investor can observe the stock price process S_t which is published and available to the public, however, the drift process μ_t and the information of Brownian motions $(W_t)_{0\leq t\leq T}$ and $(B_t)_{0\leq t\leq T}$ are unknown. We shall call this as the "*partial observations information*" scenario. This investment and consumption optimization problem with incomplete information will be modeled by requiring the investment strategy $(\pi_t)_{0\leq t\leq T}$ and consumption policy $(c_t)_{0\leq t\leq T}$ be only adapted to the partial observation filtration $\mathbb{F}^S = (\mathcal{F}_t^S)_{0\leq t\leq T}$ where $\mathcal{F}_t^S = \sigma\{S_u : 0 \leq u \leq t\}$, which is strictly smaller than the background full information $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$.

Applying the famous Kalman-Bucy stochastic filtering theorem, we can first define the **Innovation Process** in our market model as:

$$d\hat{W}_t \triangleq \frac{1}{\sigma_S}\Big[(\mu_t - \hat{\mu}_t)dt + \sigma_S dW_t\Big] = \frac{1}{\sigma_S}\Big(\frac{dS_t}{S_t} - \hat{\mu}_t dt\Big), \quad 0 \leq t \leq T, \tag{3.1}$$

which is a Brownian motion under the partial observations filtration $\mathcal{F}_t^S$, where the process $\hat{\mu}_t = \mathbb{E}\Big[\mu_t\Big|\mathcal{F}_t^S\Big]$ is the conditional estimation of drift process μ_t.

Moreover, by the same Kalman-Bucy filtering theorem, the process $\hat{\mu}_t$ satisfies the linear SDE:

$$\begin{aligned} d\hat{\mu}_t =& -\lambda(\hat{\mu}_t-\bar{\mu})dt+\Big(\frac{\hat{\Omega}_t+\sigma_S\sigma_\mu\rho}{\sigma_S}\Big)d\hat{W}_t, \\ =&\Big(-\lambda-\frac{\Big(\hat{\Omega}_t+\sigma_S\sigma_\mu\rho\Big)}{\sigma_S^2}\Big)\hat{\mu}_t dt+\lambda\bar{\mu}dt+\frac{\Big(\hat{\Omega}_t+\sigma_S\sigma_\mu\rho\Big)}{\sigma_S^2}\frac{dS_t}{S_t}, \quad 0\le t\le T, \end{aligned} \tag{3.2}$$

with $\hat{\mu}_0 = \mathbb{E}\Big[\mu_0\Big|\mathcal{F}_0^S\Big] = \eta_0$. So we can solve for $\hat{\mu}_t$ as the strong solution of the SDE (3.2) by knowing the stock price process S_t and $\hat{\Omega}_t$.

And the conditional variance $\hat{\Omega}_t = \mathbb{E}\Big[(\mu_t-\hat{\mu}_t)^2\Big|\mathcal{F}_t^S\Big]$ satisfies the deterministic Riccati ODE:

$$d\hat{\Omega}_t = \Big[-\frac{1}{\sigma_S^2}\hat{\Omega}_t^2+\Big(-\frac{2\sigma_\mu\rho}{\sigma_S}-2\lambda\Big)\hat{\Omega}_t+(1-\rho^2)\sigma_\mu^2\Big]dt, \quad 0\le t\le T, \tag{3.3}$$

with $\hat{\Omega}(0) = \mathbb{E}\Big[(\mu_0-\eta)^2\Big|\mathcal{F}_0^S\Big] = \theta_0$, which has an explicit solution as:

$$\hat{\Omega}_t = \hat{\Omega}(t;\theta_0) = \sqrt{k}\sigma_S\frac{k_1\exp(2(\frac{\sqrt{k}}{\sigma_S})t)+k_2}{k_1\exp(2(\frac{\sqrt{k}}{\sigma_S})t)-k_2}-(\lambda+\frac{\sigma_\mu\rho}{\sigma_S})\sigma_S^2, \quad 0\le t\le T, \tag{3.4}$$

where:

$$\begin{aligned} k &= \lambda^2\sigma_S^2 S+2\sigma_S\sigma_\mu\lambda\rho+\sigma_\mu^2, \\ k_1 &= \sqrt{k}\sigma_S+(\lambda\sigma_S^2+\sigma_S\sigma_\mu\rho)+\theta_0, \\ k_2 &= -\sqrt{k}\sigma_S+(\lambda\sigma_S^2+\sigma_S\sigma_\mu\rho)+\theta_0. \end{aligned}$$

By simple observation, we see $\hat{\Omega}(t)$ converges monotonically to the value

$$\theta^* = \sigma_S\sqrt{\lambda^2\sigma_S^2+2\sigma_S\sigma_\mu\lambda\rho+\sigma_\mu^2}-(\lambda\sigma_S^2+\sigma_S\sigma_\mu\rho)>0 \tag{3.5}$$

as time $t\to+\infty$, which we call as "steady state learning" (see also *Brennan* [3]). This convergence property of $\hat{\Omega}(t)$ tells us the precision of the drift estimate goes from an initial condition to a steady state in the long time run, and after large time T, new return observations contribute to updating the estimated value of the state variable, but seldom reduce the variance of the estimation error. More precisely, by the evolution of Riccati ODE (3.3), we have the monotone solution $\hat{\Omega}(t)$ on $(0,\infty)$ has the bounds

$$\min(\theta_0,\theta^*)\le\hat{\Omega}(t)\le\max(\theta_0,\theta^*), \quad \forall t\ge 0. \tag{3.6}$$

Under the observation filtration $(\mathcal{F}_t^S)_{0\le t\le T}$, we can instead rewrite stock price dynamics (2.1) driven by the innovation process $\hat{W}_t$ as:

$$dS_t = \hat{\mu}_t S_t dt+\sigma_S S_t d\hat{W}_t, \quad 0\le t\le T. \tag{3.7}$$

Notice we are now seeking the optimal investment and consumption strategies π_t and c_t which are only progressively measurable with respect to the partial observations filtration $\mathcal{F}_t^S$, where the

standard of living process Z_t satisfies the ODE

$$dZ_t = (\delta(t)c_t - \alpha(t)Z_t)dt, \quad 0 \le t \le T, \tag{3.8}$$

and under the partial observations filtration $\mathcal{F}_t^S$, the wealth process dynamics (2.4) under π_t and c_t will be rewritten as:

$$dX_t = (\pi_t\hat{\mu}_t - c_t)dt + \sigma_S\pi_t d\hat{W}_t, \quad 0 \le t \le T. \tag{3.9}$$

Our goal now is to maximize the consumption with linear habit formation and terminal wealth by power utility preference under the partial observations filtration $\mathcal{F}_t^S$:

$$v(x_0, z_0, \eta_0, \theta_0) = \sup_{\pi, c\in\mathcal{A}} \mathbb{E}\Big[\int_0^T \frac{(c_s - Z_s)^p}{p} ds + \frac{(X_T)^p}{p}\Big], \tag{3.10}$$

where we take the risk aversion coefficient $p < 1$ and $p \neq 0$.

Our aim is to provide an analytic solution of the control problem (3.10) using direct dynamic programming, i.e., first solve the Dynamic Programming equation analytically and then perform a rigorous verification argument. Therefore, there is no need to either define the value function at later times or to prove the Dynamic Programming Principle involving some complicated measurable selection arguments.

Formally, at this level, we look for a smooth function $\tilde{v}(t, x, z, \eta, \theta)$ defined on an appropriate domain such that the process

$$\int_0^t \frac{(c_s - Z_s)^p}{p} ds + \tilde{v}(t, X_t, Z_t, \hat{\mu}_t, \hat{\Omega}_t), \quad \forall t \in [0, T],$$

is a local supermartingale for each admissible control $(\pi_t, c_t) \in \mathcal{A}$ and a local martingale for the optimal feedback control $(\pi_t^*, c_t^*) \in \mathcal{A}$. The appropriate domain will be carefully defined later after we solve the associated HJB equation explicitly, moreover, some financial intuitions will also be clarified based on the domain of the solution.

Furthermore, we recall that the conditional variance process $\hat{\Omega}_t = \hat{\Omega}(t, \theta_0)$ is actually a deterministic function of time explicitly given by (3.4). We can therefore set the variable θ in the definition of $\hat{v}$ by a deterministic function $\theta = \theta(t, \theta_0)$ depending on the parameter θ_0 to reduce the dimension of the function $\tilde{v}$, i.e., the variable $\theta(t; \theta_0)$ is absorbed by the variable t. Hence, we can define the function $V(t, x, z, \eta; \theta_0)$ as

$$V(t, x, z, \eta; \theta_0) \triangleq \tilde{v}(t, x, z, \eta, \theta(t, \theta_0)),$$

and our target above can be simplified into finding a smooth enough function $V(t, x, z, \eta; \theta_0)$ on some appropriate domain , denoted by $V(t, x, z, \eta)$ for simplicity, such that

$$\int_0^t \frac{(c_s - Z_s)^p}{p} ds + V(t, X_t, Z_t, \hat{\mu}_t), \quad \forall t \in [0, T],$$

is a local supermartingale for each admissible control $(\pi_t, c_t) \in \mathcal{A}$ and a local martingale for the optimal feedback control $(\pi_t^*, c_t^*) \in \mathcal{A}$, for each fixed initial value $\hat{\Omega}(0) = \theta_0$.

An investment and consumption pair process (π_t, c_t) is said in the **Admissible Control Space**

$\mathcal{A}$: if it is $\mathcal{F}_t^S$-progressively measurable, and satisfies the integrability conditions:

$$\int_0^T \pi_t^2 dt < +\infty, \quad a.s. \quad \text{and} \quad \int_0^T c_t dt < +\infty, \quad a.s. \tag{3.11}$$

with the addictive habits constraint that: $c_t \geq Z_t, \ \forall t \in [0,T]$. Moreover, no bankruptcy is allowed, i.e., the investor's wealth remains nonnegative: $X_t \geq 0, \ 0 \leq t \leq T$.

By the definition of $V(t,x,z,\eta)$ and Itô's formula, we can formally derive the Hamilton-Jacobi-Bellman (HJB) equation as:

$$\begin{aligned} & V_t - \alpha(t)zV_z - \lambda(\eta-\bar{\mu})V_\eta + \frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}V_{\eta\eta} + \max_{c\in\mathcal{A}}\Big[-cV_x + c\delta(t)V_z \\ & + \frac{(c-z)^p}{p}\Big] + \max_{\pi\in\mathcal{A}}\Big[\pi\eta V_x + \frac{1}{2}\sigma_S^2\pi^2 V_{xx} + V_{x\eta}\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)\pi\Big] = 0, \end{aligned} \tag{3.12}$$

with the terminal condition $V(T,x,z,\eta) = \frac{x^p}{p}$.

3.2. **The Decoupled Reduced Form Solution.**

If $V(t,x,z,\eta)$ is smooth enough, the first order condition formally derives

$$\begin{aligned} \pi^*(t,x,z,\eta) &= \frac{-\eta V_x - \Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)V_{x\eta}}{\sigma_S^2 V_{xx}}, \\ c^*(t,x,z,\eta) &= z + \Big(V_x - \delta(t)V_z\Big)^{\frac{1}{p-1}}. \end{aligned} \tag{3.13}$$

which achieve the maximum over control policies π and c respectively.

Plugging forms of (3.13) for π^* and c^*, the HJB equation becomes:

$$\begin{aligned} & V_t - \alpha(t)zV_z - \lambda(\eta-\bar{\mu})V_\eta + \frac{(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{2\sigma_S^2}V_{\eta\eta} - \frac{\eta(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)}{\sigma_S^2}\frac{V_xV_{x\eta}}{V_{xx}} \\ & -\frac{\eta^2}{2\sigma_S^2}\frac{V_x^2}{V_{xx}} - \frac{(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{2\sigma_S^2}\frac{V_{x\eta}^2}{V_{xx}} - z(V_x-\delta(t)V_z) - \frac{p-1}{p}\Big(V_x-\delta(t)V_z\Big)^{\frac{p}{p-1}} = 0. \end{aligned} \tag{3.14}$$

We expect that the smooth solution $V(t,x,z,\eta)$ of the HJB equation at time $t=0$ is actually the value function, i.e., $V(0,x_0,z_0,\eta_0;\theta_0) = v(x_0,z_0,\eta_0,\theta_0)$. Due to the homogeneity property of the power utility function and the linearity of dynamics (3.9) and (3.8) for X_t and Z_t respectively, it's easy to see that if $V(t,x,z,\eta)$ is finite, then it is homogeneous in (x,z) with degree p, i.e., for any $x>0$, $z\geq 0$ and the positive constant k, we have $V(t,kx,kz,\eta) = k^pV(t,x,z,\eta)$. It therefore makes sense for us to seek the value function of the form:

$$V(t,x,z,\eta) = \frac{\Big[(x-m(t,\eta)z)\Big]^p}{p}M(t,\eta)$$

for some test functions $m(t,\eta)$ and $M(t,\eta)$ to be determined. By the virtue of $V(T) = \frac{x^p}{p}$, we will require $M(T,\eta)=1$ and $m(T,\eta)=0$.

After we do the direct substitution in the above Equation (3.14) and divide the equation on both

sides by $(x-m(t,\eta)z)^p$, the HJB equation becomes

$$\frac{\Big[f(t,m)z+\lambda(\eta-\bar{\mu})m_\eta-\frac{1}{2\sigma_S^2}(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2m_{\eta\eta}+\frac{\eta}{\sigma_S^2}(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)m_\eta\Big]}{x-m(t,\eta)z}M$$
$$+\frac{1}{p}M_t-\frac{\lambda(\eta-\bar{\mu})}{p}M_\eta+\frac{(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{2p\sigma_S^2}M_{\eta\eta}-\frac{\eta(\hat{\Omega}(t)+\sigma_S\sigma_S\rho)}{(p-1)\sigma_S^2}M_\eta \tag{3.15}$$
$$-\frac{\eta^2}{2(p-1)\sigma_S^2}M-\frac{(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{2(p-1)\sigma_S^2}\frac{M_\eta^2}{M}-\frac{p-1}{p}\Big(1+\delta(t)m(t,\eta)\Big)^{\frac{p}{p-1}}M^{\frac{p}{p-1}}=0.$$

where we set

$$f(t,m)=-m_t+\alpha(t)m-(1+\delta(t)m).$$

Since the Equation (3.15) above holds for all values of x and z, we can naturally set the unknown priori function $m(t,\eta)=m(t)$ as a deterministic function in time t and independent of the variable η which satisfies:

$$f(t,m)=-m_t(t)+\alpha(t)m(t)-(1+\delta(t)m(t))=0 \tag{3.16}$$

with the terminal condition $m(T)=0$, which is equivalent to:

$$m(t)=\int_t^T\exp\Big(\int_t^s(\delta(v)-\alpha(v))dv\Big)ds.\quad 0\le t\le T. \tag{3.17}$$

Now we can substitute the function $m(t)$ into the equation (3.15) above, and simplify it as:

$$M_t+\frac{p\eta^2}{2(1-p)\sigma_1^2}M+\frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}M_{\eta\eta}+(1-p)\Big(1+\delta(t)m(t)\Big)^{\frac{p}{p-1}}M^{\frac{p}{p-1}}$$
$$+\Big[-\lambda(\eta-\bar{\mu})+\frac{\eta(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)p}{(1-p)\sigma_S^2}\Big]M_\eta+\frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2p}{2(1-p)\sigma_S^2}\frac{M_\eta^2}{M}=0. \tag{3.18}$$

Now in order to solve the above nonlinear PDE (3.18), we can set the power transform as

$$M(t,\eta)=N(t,\eta)^{1-p} \tag{3.19}$$

This idea of power transform was first introduced in *Zariphopoulou* [24].

And the nonlinear PDE (3.18) for $M(t,\eta)$ reduces to the linear parabolic PDE for $N(t,\eta)$ as:

$$N_t+\frac{p\eta^2}{2(1-p)^2\sigma_S^2}N(t,\eta)+\frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}N_{\eta\eta}+\Big(1+\delta(t)W(t)\Big)^{\frac{p}{p-1}}$$
$$+\Big[-\lambda(\eta-\bar{\mu})+\frac{\eta\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)p}{(1-p)\sigma_S^2}\Big]N_\eta(t,\eta)=0 \tag{3.20}$$

with $N(T,\eta)=1$.

For the above linear PDE (3.20) of $N(t,\eta)$, we can further solve it explicitly as:

$$\begin{aligned} N(t,\eta) = &\int_t^T \Big(1+\delta(s)W(s)\Big)^{\frac{p}{p-1}} \exp\Big(A(s,t)\eta^2 + B(s,t)\eta + C(s,t)\Big) ds \\ &+ \exp\Big(A(T,t)\eta^2 + B(T,t)\eta + C(T,t)\Big), \end{aligned} \tag{3.21}$$

where we have for $0 \le t \le s \le T$, $A(s,t) = A(t;s)$, $B(s,t) = B(t;s)$ and $C(s,t) = C(t;s)$ satisfying the following ODEs:

$$A_t(t) + \frac{p}{2(1-p)^2\sigma_S^2} + 2\Big[-\lambda + \frac{p(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)}{\sigma_S^2(1-p)}\Big]A(t) + \frac{2(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}A^2(t) = 0; \tag{3.22}$$

$$B_t(t) + \Big[-\lambda + \frac{p(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)}{\sigma_S^2(1-p)}\Big]B(t) + 2\lambda\bar{\mu}A(t) + \frac{2(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}A(t)B(t) = 0; \tag{3.23}$$

$$C_t(t) + \lambda\bar{\mu}B(t) + \frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}\Big(B^2(t) + 2A(t)\Big) = 0; \tag{3.24}$$

with terminal conditions: $A(s) = B(s) = C(s) = 0$.

We remark that the above ODEs are similar to the ODEs obtained by *Brendle* [2] for terminal wealth optimization problem with partial observations, and he made an insightful observation that we can actually solve the above 3 ODEs with time t dependent coefficients by solving the following 5 auxiliary ODEs with constant coefficients, see section 4 of *Brendle* [2] for the detail proof.

Theorem 3.1. *For $0 \le t \le s \le T$, consider the following auxiliary ODEs for $a(t)$, $b(t)$, $c(t)$, $f(t)$ and $g(t)$:*

$$a_t(t) = -\frac{2(1-p+p\rho^2)}{1-p}\sigma_\mu^2 a^2(t) + \Big(2\lambda - \frac{2p\rho\sigma_\mu}{(1-p)\sigma_S}\Big)a(t) - \frac{p}{2(1-p)\sigma_S^2}, \tag{3.25}$$

$$b_t(t) = -\frac{2(1-p+p\rho^2)}{1-p}\sigma_\mu^2 a(t)b(t) - 2\lambda\bar{\mu}a(t) + \Big(\lambda - \frac{p\rho\sigma_\mu}{(1-p)\sigma_S}\Big)b(t), \tag{3.26}$$

$$c_t(t) = -\sigma_\mu^2 a(t) - \frac{(1-p+p\rho^2)\sigma_\mu^2}{2(1-p)}b^2(t) - \lambda\bar{\mu}b(t), \tag{3.27}$$

$$f_t(t) = -2(1-\rho^2)\sigma_\mu^2 f^2(t) + 2\frac{\lambda\sigma_S + \rho\sigma_\mu}{\sigma_S}f(t) + \frac{1}{2\sigma_S^2}, \tag{3.28}$$

$$g_t(t) = \sigma_\mu^2(1-\rho^2)(f(t) - a(t)), \tag{3.29}$$

with the terminal conditions $a(s) = b(s) = c(s) = f(s) = g(s) = 0$, and if we adopt the convention $\frac{0}{0} = 0$, then for the functions defined by:

$$\tilde{A}(t;s) = \frac{a(t)}{(1-p)\Big(1 - 2a(t)\hat{\Omega}(t)\Big)},$$

$$\tilde{B}(t;s) = \frac{b(t)}{(1-p)\Big(1-2a(t)\hat{\Omega}(t)\Big)},$$

$$\begin{aligned}\tilde{C}(t;s) =& \frac{1}{1-p}\Big[c(t) + \frac{\hat{\Omega}(t)}{\Big(1-2a(t)\hat{\Omega}(t)\Big)}b^2(t) - \frac{1-p}{2}\log\Big(1-2a(t)\hat{\Omega}(t)\Big)\\ &-\frac{p}{2}\log\Big(1-2f(t)\hat{\Omega}(t)\Big) - pg(t)\Big],\end{aligned}$$

we have the equivalence that:

$$A(t;s) = \tilde{A}(t;s),\ B(t;s) = \tilde{B}(t;s),\ C(t;s) = \tilde{C}(t;s), \quad 0 \le t \le s \le T. \tag{3.30}$$

Remark 3.1. *The equivalence result* (3.30) *reveals that solving the ODEs* (3.22), (3.23), (3.24) *with variable coefficients is equivalent to solving the auxiliary ODEs* (3.25), (3.26), (3.27), (3.28) *and* (3.29) *with constant coefficients in such an order that we solve the Riccati ODE* (3.25) *first, and substitute the solution $a(t;s)$ into ODE* (3.26) *and solve for the solution $b(t;s)$, and etc.*

Actually, we can find fully explicit forms for $a(t;s)$, $b(t;s)$, $c(t;s)$, $f(t;s)$ and $g(t;s)$. We list all four different cases of fully explicit solutions in the Appendix depending on the risk aversion coefficient p and the market coefficients σ_S, σ_μ, λ and ρ. By simple substitutions, we can therefore solve the ODEs (3.22), (3.23), (3.24) *for $A(t;s)$, $B(t;s)$ and $C(t;s)$ fully explicitly.*

Now, for $t \in [0,T]$, $\eta \in (-\infty,+\infty)$, we can define the *effective domain* for the pair (x,z) as:

$$(x,z) \in \mathbb{D}_t = \{(x',z') \in (0,+\infty)\times[0,+\infty);\ x' \ge m(t)z'\},\ 0 \le t \le T, \tag{3.31}$$

and the function

$$\begin{aligned}V(t,x,z,\eta) =& \Big[\int_t^T \Big(1+\delta(s)m(s)\Big)^{\frac{p}{p-1}}\exp\Big(A(s,t)\eta^2 + B(s,t)\eta + C(s,t)\Big)ds\\ &+\exp\Big(A(T,t)\eta^2 + B(T,t)\eta + C(T,t)\Big)\Big]^{1-p}\frac{\Big[x-m(t)z\Big]^p}{p}\end{aligned} \tag{3.32}$$

is well defined on $[0,T]\times\mathbb{D}_t\times\mathbb{R}$ and it's the classical solution of the HJB equation (3.12), where $m(t) = \int_t^T \exp(\int_t^s(\delta(v)-\alpha(v))dv)ds$, and $A(s,t)$, $B(s,t)$, $C(s,t)$ are solutions of ODEs (3.22), (3.23), (3.24).

Remark 3.2. *In our main result below, we want to verify that the above classical solution $V(t,x,z,\eta)$ at time $t=0$ equals our primal value function defined in* (3.10), *i.e., $V(0,x_0,z_0,\eta_0;\theta_0) = v(x_0,z_0,\eta_0,\theta_0)$. However, the effective domain of $V(t,x,z,\eta)$ motivates some constraints on the optimal wealth process X_t^* and habit formation process Z_t^*. To wit, function $V(t,x,z,\eta) = -\infty$ when $x < m(t)z$, which mandates that $X_t^* \ge m(t)Z_t^*$ for each $t \in [0,T]$ to ensure the process $V(t,X_t^*,Z_t^*,\hat{\mu}_t)$ is well defined. In particular, when $t=0$, we have to mandate the initial wealth-habit budget constraint that $x_0 \ge m(0)z_0$.*

3.3. **The Main Result.**

Theorem 3.2 (The Verification Theorem)**.**
Build upon the initial wealth-habit budget constraint $x_0 > m(0)z_0$, then either if risk aversion

constant

$$p < 0;$$

or if

$$0 < p < 1, \qquad \textit{together with market coefficients } \sigma_S, \sigma_\mu, \lambda, \Theta, \rho$$

satisfy the additional assumption that explicit functions $a(t;s)$, $b(t;s)$, $c(t;s)$, $f(t;s)$ *and* $g(t;s)$ *defined in Theorem* 3.1 *are bounded and* $1 - a(t;s)\hat{\Omega}(t) \neq 0$ *on* $0 \leq t \leq s \leq T$ *(See Appendix A for the detail discussion). Moreover, we assume*

$$\frac{p(1+p)}{(1-p)^2} < \frac{\lambda^2 \sigma_S^4}{4(\Theta + \sigma_S\sigma_\mu\rho)^2}, \tag{3.33}$$

where $\Theta \triangleq \max\{\theta_0,\ \theta^*\}$ *and* θ^* *is define in* (3.5)*. And the upper bound* $\bar{K}_1$ *of* $A(t;s)$ *on* $0 \leq t \leq s \leq T$ *satisfies*

$$4\bar{K}_1 < \frac{\lambda\sigma_S^2}{(\Theta + \sigma_S\sigma_\mu\rho)^2}. \tag{3.34}$$

Then, the solution (3.32) *of HJB equation equals the value function defined in* (3.10)*:*

$$V(0, x_0, z_0, \eta_0; \theta_0) = v(x_0, z_0, \eta_0, \theta_0). \tag{3.35}$$

And the optimal investment policy π_t^* *and optimal consumption policy* c_t^* *are given in the feedback form:* $\pi_t^* = \pi^*(t, X_t^*, Z_t^*, \hat{\mu}_t)$ *and* $c_t^* = c^*(t, X_t^*, Z_t^*, \hat{\mu}_t)$, $0 \leq t \leq T$, *where the function* $\pi^*(t,x,z,\eta) : [0,T] \times \mathbb{D}_t \times \mathbb{R} \longrightarrow \mathbb{R}$ *is defined by:*

$$\pi^*(t,x,z,\eta) = \Big[\frac{\eta}{(1-p)\sigma_S^2} + \frac{\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)}{\sigma_S^2}\frac{N_\eta(t,\eta)}{N(t,\eta)}\Big](x - m(t)z),\ \ 0 \leq t \leq T. \tag{3.36}$$

$c^*(t,x,z,\eta) : [0,T] \times \mathbb{D}_t \times \mathbb{R} \longrightarrow \mathbb{R}_+$ *is defined by:*

$$c^*(t,x,z,\eta) = z + \frac{(x - m(t)z)}{\Big(1 + \delta(t)m(t)\Big)^{\frac{1}{1-p}} N(t,\eta)}, \quad 0 \leq t \leq T. \tag{3.37}$$

And the optimal wealth process X_t^*, *for* $0 \leq t \leq T$, *is given explicitly by:*

$$X_t^* = (x_0 - m(0)z_0)\frac{N(t,\hat{\mu}_t)}{N(0,\eta)}\exp\Big(\int_0^t \frac{(\hat{\mu}_u)^2}{2(1-p)\sigma_S^2}du + \int_0^t \frac{\hat{\mu}_u}{(1-p)\sigma_S}d\hat{W}_u\Big) + m(t)Z_t^*, \tag{3.38}$$

where $m(t)$ *and* $N(t,\eta)$ *are defined in* (3.17) *and* (3.21) *respectively.*

Remark 3.3. *The more complex structure of feedback forms of optimal investment and consumption policies is the consequence of the time non-separability of the instantaneous utility with habit formation. We can see the portfolio/wealth ratio* $\frac{\pi^*}{X^*}$ *and consumption/wealth ratio* $\frac{c^*}{X^*}$ *are now depending on the habit-formation/wealth ratio* $\frac{Z^*}{X^*}$*:*

$$\frac{\pi^*}{X^*} = \Big[\frac{\hat{\mu}_t}{(1-p)\sigma_S^2} + \frac{\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)}{\sigma_S^2}\frac{N_\eta(t,\hat{\mu}_t)}{N(t,\hat{\mu}_t)}\Big](1 - m(t)\frac{Z^*}{X^*}),$$

and

$$\frac{c^*}{X^*} = \frac{1}{\big(1+\delta(t)m(t)\big)^{\frac{1}{1-p}}N(t,\hat{\mu}_t)} + \Big(1 - \frac{m(t)}{\big(1+\delta(t)m(t)\big)^{\frac{1}{1-p}}N(t,\hat{\mu}_t)}\Big)\frac{Z^*}{X^*},$$

Moreover, although function $c^*(\cdot,x,\cdot,\cdot)$ remains still linear and increasing in $x>0$, $c^*(\cdot,\cdot,z,\cdot)$ is not necessary increasing in $z\geq 0$, which shows the increase of initial habit dose not necessarily imply the increase of optimal consumption stream. And since the dependence of $c^*(t,x,z,\eta)$ on the discounting factors $\alpha(t)$ and $\delta(t)$ are even more complicated, the optimal consumption process c_t^* is not necessarily monotone in the habit formation process Z_t^*.

4. Proof of The Verification Theorem

We will first show the consumption constraint $c_t \geq Z_t$ implies the constraint on the controlled wealth process by the following proposition:

Proposition 4.1. *The admissible space $\mathcal{A}$ is not empty if and only if the initial budget constraint with habit formation $x_0 \geq m(0)z_0$ is fulfilled. Moreover, for each pair of investment and consumption policy $(\pi,c)\in\mathcal{A}$, the controlled wealth process $X_t^{\pi,c}$ satisfies the constraint:*

$$X_t^{\pi,c} \geq m(t)Z_t, \quad 0\leq t\leq T, \tag{4.1}$$

where the deterministic function $m(t)$ is defined in (3.16) *and refers to the cost of subsistence consumption per unit of standard of living at time t.*

Proof. On one hand, let's assume $x_0 \geq m(0)z_0$, then we can always take $\pi_t \equiv 0$, and $c_t = z_0 \exp\Big(\int_0^t(\delta(v)-\alpha(v))dv\Big)$ for $t\in[0,T]$, it is easy to verify $X_t^{\pi,c}\geq 0$ and $c_t \equiv Z_t$ so that $(\pi,c)\in\mathcal{A}$, and hence $\mathcal{A}$ is not empty.

On the other hand, starting from $t=0$ with the wealth x_0 and the standard of living z_0, the addictive habits constraint $c_t \geq Z_t,\ 0\leq t\leq T$ implies the consumption must always exceed the *subsistence consumption* $\bar{c}_t = Z(t;\bar{c}_t)$ which satisfies

$$d\bar{c}_t = (\delta(t)-\alpha(t))\bar{c}_t dt, \quad \bar{c}_0 = z_0, \quad 0\leq t\leq T, \tag{4.2}$$

Indeed, we first recall by the definition of Z_t that $dZ_t = (\delta_t c_t - \alpha_t Z_t)dt$ with $Z_0 = z\geq 0$, and the constraint that $c_t\geq Z_t$ implies

$$dZ_t \geq (\delta_t Z_t - \alpha_t Z_t)dt, \quad Z_0 = z_0. \tag{4.3}$$

By the simple subtraction of (4.3) and (4.2), one can get

$$d(Z_t-\bar{c}_t) \geq (\delta_t-\alpha_t)(Z_t-\bar{c}_t)dt, \quad Z_0-\bar{c}_0 = 0,$$

from which we can derive that

$$e^{\int_0^t(\delta_s-\alpha_s)ds}(Z_t-\bar{c}_t)\geq 0, \quad \forall t\in[0,T]. \tag{4.4}$$

And hence we can obtain $c_t \geq \bar{c}_t$, which is equivalent to

$$c_t \geq z_0 \exp\Big(\int_0^t (\delta(v) - \alpha(v))dv\Big), \quad 0 \leq t \leq T. \tag{4.5}$$

Define the exponential local martingale

$$\widetilde{H}_t = \exp\Big(-\int_0^t \frac{\hat{\mu}_v}{\sigma_S} d\hat{W}_v - \frac{1}{2}\int_0^t \frac{\hat{\mu}_v^2}{\sigma_S^2} dv\Big), \quad 0 \leq t \leq T. \tag{4.6}$$

Since $\hat{\mu}_t$ follows the dynamics (3.2), which is

$$\hat{\mu}_t = e^{-t\lambda}\eta + \bar{\mu}(1 - e^{-t\lambda}) + \int_0^t e^{\lambda(u-t)} \frac{\Big(\hat{\Omega}(u) + \sigma_S \sigma_\mu \rho\Big)}{\sigma_S} d\hat{W}_u.$$

similar to the proofs of Corollary 3.5.14 and Corollary 3.5.16 in *Karatzas and Shreve* [12], Beneš' condition implies $\widetilde{H}$ is a true martingale with respect to $(\Omega, \mathcal{F}^S, \mathbb{P})$,

Now define the probability measure $\widetilde{\mathbb{P}}$ as

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \widetilde{H}_T,$$

Girsanov theorem states that

$$\widetilde{W}_t \triangleq \hat{W}_t + \int_0^t \frac{\hat{\mu}_v}{\sigma_S} dv, \quad 0 \leq t \leq T$$

is a Brownian motion under $(\widetilde{\mathbb{P}}, (\mathcal{F}_t^S)_{0 \leq t \leq T})$.

Then we can rewrite the wealth process dynamics as:

$$X_T + \int_0^T c_v dv = x + \int_0^T \pi_v \sigma_S d\widetilde{W}_v,$$

Since we have $X_T \geq 0$, it's easy to see that $\int_0^t \pi_v \sigma_S d\widetilde{W}_v$ is a supermartingale under $(\Omega, \mathbb{F}^S, \widetilde{\mathbb{P}})$, and take the expectation under $\widetilde{\mathbb{P}}$, we have:

$$x_0 \geq \widetilde{\mathbb{E}}\Big[\int_0^T c_v dv\Big].$$

Follow the inequality (4.5), we will further have:

$$x_0 \geq z_0 \widetilde{\mathbb{E}}\Big[\int_0^T \exp\Big(\int_0^v (\delta(u) - \alpha(u))du\Big)dv\Big].$$

Since $\delta(t)$ and $\alpha(t)$ are deterministic functions, we easily arrive $x_0 \geq m(0)z_0$.

In general, for $\forall t \in [0, T]$, follow the same procedure, we can then take conditional expectation under filtration $\mathcal{F}_t^S$, and get

$$X_t \geq Z_t \widetilde{\mathbb{E}}\Big[\int_t^T \exp\Big(\int_t^v (\delta(u) - \alpha(u))du\Big)dv \Big| \mathcal{F}_t^S\Big],$$

again since $\delta(t), \alpha(t)$ are deterministic, we obtain $X_t \geq m(t)Z_t, \quad 0 \leq t \leq T.$ □

Remark 4.1. *The constraint on the controlled wealth process X_t and the habit formation process Z_t agrees with the effective domain $\{(x, z) \in (0, \infty) \times [0, \infty) : x \geq m(t)z\}$ of the HJB equation for*

the values of x and z. Aside from the consequence that the process $V(t, X_t, Z_t, \hat{\mu}_t)$ is therefore well defined, it plays a critical role in our following proof of the verification lemma.

4.1. **The Case** $p < 0$.

PROOF OF THEOREM 3.2.

First, for any pair of admissible control $(\pi_t, c_t) \in \mathcal{A}$, Itô's lemma gives

$$d\Big[V(t, X_t, Z_t, \hat{\mu}_t)\Big] = \Big[\mathcal{G}^{\pi_t,c_t} V(t, X_t, Z_t, \hat{\mu}_t)\Big] dt + \Big[V_x \sigma_S \pi_t + V_\eta \frac{\Big(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho\Big)}{\sigma_S}\Big] d\hat{W}_t, \tag{4.7}$$

where we define the process $\mathcal{G}^{\pi_t,c_t} V(t, X_t, Z_t, \hat{\mu}_t)$ as

$$\begin{aligned}\mathcal{G}^{\pi_t,c_t} V(t, X_t, Z_t, \hat{\mu}_t) = {} & V_t - \alpha(t) Z_t V_z - \lambda(\hat{\mu}_t - \bar{\mu}) V_\eta + \frac{\Big(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho\Big)^2}{2\sigma_S^2} V_{\eta\eta} \\ & - c_t V_x + c_t \delta(t) V_z + \frac{(c_t - Z_t)^p}{p} + \pi_t \hat{\mu}_t V_x + \frac{1}{2} \sigma_S^2 \pi_t^2 V_{xx} + V_{x\eta} \Big(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho\Big) \pi_t.\end{aligned}$$

Recall $V(t, x, z, \eta)$ is the classical solution of HJB equation (3.12), choose the localizing sequence τ_n, we integrate the equation (4.7) on $[0, \tau_n \wedge T]$, and take the expectation, we have

$$V(0, x_0, z_0, \eta_0) \geq \mathbb{E}\Big[\int_0^{\tau_n \wedge T} \frac{(c_s - Z_s)^p}{p} ds\Big] + \mathbb{E}\Big[V(\tau_n \wedge T, X_{\tau_n \wedge T}, Z_{\tau_n \wedge T}, \hat{\mu}_{\tau_n \wedge T})\Big]. \tag{4.8}$$

Now, we follow the idea by *Janeček and Sîrbu* [11], let's fix this pair of control choice $(\pi_t, c_t) \in \mathcal{A} = \mathcal{A}_{x_0}$, where we denote $\mathcal{A}_{x_0}$ as the admissible space with initial endowment x_0. And for $\forall \epsilon > 0$, it is clear that $\mathcal{A}_{x_0} \subseteq \mathcal{A}_{x_0+\epsilon}$, and $(\pi_t, c_t) \in \mathcal{A}_{x_0+\epsilon}$. Also it is clear that $X_t^{x_0+\epsilon} = X_t^{x_0} + \epsilon = X_t + \epsilon, \quad 0 \leq t \leq T$. Follow the same procedure above, and notice process Z_t keeps the same under the consumption policy c_t, then we can obtain:

$$\begin{aligned}V(0, x_0 + \epsilon, z_0, \eta_0) \geq {} & \mathbb{E}\Big[\int_0^{\tau_n \wedge T} \frac{(c_s - Z_s)^p}{p} ds\Big] \\ & + \mathbb{E}\Big[V(\tau_n \wedge T, X_{\tau_n \wedge T} + \epsilon, Z_{\tau_n \wedge T}, \hat{\mu}_{\tau_n \wedge T})\Big].\end{aligned} \tag{4.9}$$

By Monotone Convergence Theorem, we first know:

$$\lim_{n \to +\infty} \mathbb{E}\Big[\int_0^{\tau_n \wedge T} \frac{(c_s - Z_s)^p}{p} ds\Big] = \mathbb{E}\Big[\int_0^{T} \frac{(c_s - Z_s)^p}{p} ds\Big]. \tag{4.10}$$

For simplicity, let's denote $Y_t = \Big(X_t - m(t) Z_t\Big)$, we know by definition (3.32) that:

$$V(\tau_n \wedge T, X_{\tau_n \wedge T} + \epsilon, Z_{\tau_n \wedge T}, \hat{\mu}_{\tau_n \wedge T}) = \frac{1}{p} (Y_{\tau_n \wedge T} + \epsilon)^p N_{\tau_n \wedge T}^{1-p}.$$

Proposition 4.1 gives $X_t \geq m(t) Z_t$ for $0 \leq t \leq T$ under any admissible control pair (π_t, c_t), we know $Y_{\tau_n \wedge T} + \epsilon \geq \epsilon > 0, \quad \forall 0 \leq t \leq T$. Since also $p < 0$, we will have

$$\sup_n (Y_{\tau_n \wedge T} + \epsilon)^p < \epsilon^p < +\infty. \tag{4.11}$$

Now from Remark *A*.1, we already derived that $A(t; s) \leq 0, \ \forall 0 \leq t \leq s \leq T$. Combining this with the fact that $m(s)$, $\delta(s)$ are continuous and hence bounded on $[0, T]$ and when $p < 0$, we also

have $1-a(t;s)\hat{\Omega}(t)>0$ and $1-f(t;s)\hat{\Omega}(t)>0$ as well as $a(t;s)$, $b(t;s)$, $c(t;s)$, $f(t;s)$ and $g(t;s)$ are all bounded for $0\leq t\leq s\leq T$, we deduce that the explicit solutions $B(t;s)$ and $C(t;s)$ are both bounded on $0\leq t\leq s\leq T$, hence we have:

$$N(0,\eta)\leq k_1\exp(k\eta)\ \text{ for some large constants }\ k,k_1>1,$$

which shows the existence of some constants $\bar{k},\bar{k}_1>1$ such that

$$\sup_n N^{1-p}_{\tau_n\wedge T}\leq \sup_{t\in[0,T]}\Big(k_1\exp\Big(k\hat{\mu}_t\Big)\Big)^{1-p}\leq \bar{k}_1\exp\Big(\bar{k}\sup_{t\in[0,T]}\hat{\mu}_t\Big).$$

We recall that $\hat{\mu}_t$ satisfies the Ornstein Uhlenbeck diffusion (3.2), which gives:

$$\hat{\mu}_t=e^{-t\lambda}\eta+\bar{\mu}(1-e^{-t\lambda})+\int_0^t e^{\lambda(u-t)}\frac{\Big(\hat{\Omega}(u)+\sigma_S\sigma_\mu\rho\Big)}{\sigma_S}d\hat{W}_u.$$

Hence, there exists positive constants l and $l_1>1$ large enough, such that:

$$\sup_{t\in[0,T]}\hat{\mu}_t\leq l+\sup_{t\in[0,T]}l_1\hat{W}_t,\quad t\in[0,T].$$

Using the distribution of running maximum of the Brownian motion, there exists some positive constants $\bar{l}>1$ and $\bar{l}_1$ such that

$$\mathbb{E}\Big[\sup_n N^{1-p}_{\tau_n\wedge T}\Big]\leq \bar{l}_1\mathbb{E}\Big[\exp\big(\sup_{t\in[0,T]}\bar{l}\hat{W}_t\big)\Big]<+\infty. \tag{4.12}$$

At last, by the above (4.11) and (4.12), we can conclude that

$$\mathbb{E}\Big[\sup_n V(\tau_n\wedge T,X_{\tau_n\wedge T}+\epsilon,Z_{\tau_n\wedge T},\hat{\mu}_{\tau_n\wedge T})\Big]<+\infty.$$

By virtue of Dominated Convergence Theorem, we can deduce:

$$\begin{aligned}\lim_{n\to+\infty}\mathbb{E}\Big[V(\tau_n\wedge T,X_{\tau_n\wedge T}+\epsilon,Z_{\tau_n\wedge T},\hat{\mu}_{\tau_n\wedge T})\Big]&=\mathbb{E}\Big[\frac{1}{p}(Y_T+\epsilon)^pN(T,\hat{\mu}_T)\Big]\\&=\mathbb{E}\Big[\frac{(X_T+\epsilon)^p}{p}\Big]>\mathbb{E}\Big[\frac{X_T^p}{p}\Big].\end{aligned}$$

Combine this with equation (4.9), and notice the pair of control $(\pi_t,c_t)\in\mathcal{A}$, we will see that:

$$V(0,x_0+\epsilon,z_0,\eta_0;\theta_0)\geq\sup_{\pi,c\in\mathcal{A}}\mathbb{E}\Big[\int_0^T\frac{(c_s-Z_s)^p}{p}ds+\frac{X_T^p}{p}\Big]=v(x_0,z_0,\eta_0,\theta_0).$$

Notice $V(t,x,z,\eta;\theta_0)$ is continuous in variable x, and since $\epsilon>0$ is arbitrary, we can take the limit as:

$$V(0,x_0,z_0,\eta_0;\theta_0)=\lim_{\epsilon\to 0}V(0,x_0+\epsilon,z_0,\eta_0)\geq v(x_0,z_0,\eta_0,\theta_0).$$

On the other hand, for π_t^* and c_t^* defined by (3.36) and (3.37) respectively, we first want to show the SDE for wealth process:

$$dX_t^*=(\pi_t^*\mu_t-c_t^*)dt+\sigma_S\pi_t^*d\hat{W}_t,\quad 0\leq t\leq T, \tag{4.13}$$

with initial condition $x_0>m(0)z_0$ has a unique strong solution and also satisfies $X_t^*>m(t)Z_t^*$, $\forall t\in[0,T]$.

Denote $Y_t^*=X_t^*-m(t)Z_t^*$, and apply Itô's lemma and substitute π_t^* as defined by (3.36), we

can get:

$$
\begin{aligned}
dY_t^* =& \Big[\pi_t^*\hat{\mu}_t - c_t^* - m_t(t)Z_t^* - m(t)\delta(t)c_t^* + m(t)\alpha(t)Z_t^*\Big]dt + \pi_t^*\sigma_S d\hat{W}_t \\
=& \Big[\Big(-m_t(t) + m(t)\alpha(t)\Big)Z_t^* - (1+m(t)\delta(t))c_t^* + \frac{\hat{\mu}_t^2}{(1-p)\sigma_S^2}Y_t^* \\
&+ \frac{\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)}{\sigma_S^2}\frac{N_\eta}{N}\hat{\mu}_t Y_t^*\Big]dt + \Big[\frac{\hat{\mu}_t}{(1-p)\sigma_S} + \frac{\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)}{\sigma_S}\frac{N_\eta}{N}\Big]Y_t^* d\hat{W}_t.
\end{aligned} \tag{4.14}
$$

Recall the definition of $m(t)$ by (3.16) and substitute c_t^* defined by (3.37) into (4.14) above, we will further have

$$
\begin{aligned}
dY_t^* =& \Big[-\frac{\Big(1+\delta(t)m(t)\Big)^{\frac{-p}{1-p}}}{N} + \frac{\hat{\mu}_t^2}{(1-p)\sigma_S^2} + \frac{\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)}{\sigma_S^2}\frac{N_\eta}{N}\hat{\mu}_t\Big]Y_t^* dt \\
&+ \Big[\frac{\hat{\mu}_t}{(1-p)\sigma_S} + \frac{\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)}{\sigma_S}\frac{N_\eta}{N}\Big]Y_t^* d\hat{W}_t.
\end{aligned}
$$

In order to solve X_t^* in a more explicit formula, we define the auxiliary process by:

$$
\Gamma_t = \frac{N(t,\hat{\mu}_t)}{Y_t^*}, \quad \forall 0 \le t \le T.
$$

By Itô's lemma, we can derive the SDE for process Γ_t as:

$$
\begin{aligned}
d\Gamma_t =& \frac{\Gamma_t}{N}\Big[N_t - \lambda(\hat{\mu}_t - \bar{\mu})N_\eta + \frac{\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}N_{\eta\eta} + \frac{\hat{\mu}_t\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)p}{(1-p)\sigma_S^2}N_\eta \\
&+ \Big(1+\delta(t)m(t)\Big)^{\frac{-p}{1-p}} + \frac{p\hat{\mu}_t^2}{(1-p)^2\sigma_S^2}N\Big]dt + \Gamma_t\Big[\frac{-\hat{\mu}_t}{(1-p)\sigma_S}\Big]d\hat{W}_t.
\end{aligned} \tag{4.15}
$$

Recall that $N(t,\eta)$ satisfies the linear PDE (3.21), we can simplify (4.15) to be:

$$
d\Gamma_t = \Gamma_t\Big[\frac{p\hat{\mu}_t^2}{2(1-p)^2\sigma_S^2}\Big]dt + \Gamma_t\Big[\frac{-\hat{\mu}_t}{(1-p)\sigma_S}\Big]d\hat{W}_t.
$$

Hence, we can finally get the above SDE has a unique strong solution as:

$$
\Gamma_t = \Gamma_0 \exp\Big(-\int_0^t \frac{\hat{\mu}_u^2}{2(1-p)\sigma_S^2}du - \int_0^t \frac{\hat{\mu}_u}{(1-p)\sigma_S}d\hat{W}_u\Big),
$$

Initial condition $\Gamma_0 = \frac{N(0,\eta)}{x_0 - m(0)z_0} > 0$ implies $\Gamma_t > 0, \ \forall\, 0 \le t \le T$. And, hence, we finally proved that the SDE (4.13) has a unique strong solution defined by (3.38) and the solution X_t^* satisfies the wealth process constraint (4.1)

Now, we proceed to verify π_t^* and c_t^* are actually in the admissible space $\mathcal{A}$.

First, by the definition (3.36) and (3.37), it's clear that π_t^* and c_t^* are $\mathcal{F}_t^S$ progressively measurable, and by the path continuity of $Y_t^* = X_t^* - m(t)Z_t^*$, hence, of π_t^* and c_t^*, it's easy to show that:

$$
\int_0^T (\pi_t^*)^2 dt < +\infty, \quad \text{and} \int_0^T c_t^* dt < +\infty, \quad a.s.
$$

Also, since $X_t^* > m(t)Z_t^*,\ \forall t \in [0,T]$, by the definition of c_t^*, we know the consumption constraint $c_t^* > Z_t^*,\quad \forall t \in [0,T]$ is satisfied. And hence $(\pi_t^*, c_t^*) \in \mathcal{A}$.

Given the pair of control policy (π_t^*, c_t^*) as above, following the same steps and the definition of stopping time τ_n, instead of (4.8), we can now instead get the equality:

$$V(0,x_0,z_0,\eta_0;\theta_0) = \mathbb{E}\Big[\int_0^{\tau_n\wedge T} \frac{(c_t^*-Z_t^*)^p}{p}dt\Big] + \mathbb{E}\Big[V(\tau_n\wedge T, X^*_{\tau_n\wedge T}, Z^*_{\tau_n\wedge T}, \hat{\mu}_{\tau_n\wedge T})\Big].$$

And hence, we apply Monotone Convergence Theorem again:

$$\lim_{n\to+\infty}\mathbb{E}\Big[\int_0^{\tau_n\wedge T} \frac{(c_t^*-Z_t^*)^p}{p}dt\Big] = \mathbb{E}\Big[\int_0^{T} \frac{(c_t^*-Z_t^*)^p}{p}dt\Big],$$

When $p<0$, we have function $V(t,x,z,\eta)<0$ by it's definition, and by Fatou's lemma,

$$\lim_{n\to+\infty}\sup\mathbb{E}\Big[V(\tau_n\wedge T, X^*_{\tau_n\wedge T}, Z^*_{\tau_n\wedge T}, \hat{\mu}_{\tau_n\wedge T})\Big] \le \mathbb{E}\Big[V(T, X_T^*, Z_T^*, \hat{\mu}_T)\Big] = \mathbb{E}\Big[\frac{(X_T^*)^p}{p}\Big].$$

Therefore, it gives

$$V(0,x_0,z_0,\eta_0;\theta_0) \le \mathbb{E}\Big[\int_0^{T} \frac{(c_t^*-Z_t^*)^p}{p}dt + \frac{(X_T^*)^p}{p}\Big] \le v(x_0,z_0,\eta_0,\theta_0)$$

which completes the proof. □

4.2. **The Case:** *$0<p<1$*. We proceed to prove the following two Lemmas which play important roles in the proof of the second part of our main result.

Lemma 4.1. *If constant $k>0$ satisfies:*

$$k < \frac{\lambda^2\sigma_S^2}{2(\Theta+\sigma_S\sigma_\mu\rho)^2} \tag{4.16}$$

for any $t\ge 0$, there exists a constant Λ_1 such that

$$\mathbb{E}\Big[\exp\Big(\int_0^t k\hat{\mu}_s^2 ds\Big)\Big] \le \Lambda_1 < +\infty.$$

Proof. Similar to the proof of Lemma 12 of *Fleming and Pang* [8], it is easy to choose an increasing sequence of smooth functions $Q_n(y) \nearrow ky^2$ as $n\to\infty$ such that $0\le Q_n(y)\le n$ with $\Big|Q_n'(y)\Big|$ and $\Big|Q_n''(y)\Big|$ uniformly bounded. And for each fixed $t\ge 0$ and η, we define:

$$\phi(t,\eta) = \mathbb{E}\Big[\exp\Big(\int_0^t Q_n(\hat{\mu}_s)ds\Big)\Big],$$

where $\hat{\mu}_0 = \eta$.

Similar to the proof of Feynman-Kac formula, the function $\phi(t,\eta)$ is a classical solution of the linear parabolic equation:

$$\phi_t = \frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}\phi_{\eta\eta} - \lambda(\eta-\bar{\mu})\phi_\eta + Q_n(\eta)\phi, \tag{4.17}$$

with initial condition $\phi(0,\eta)=1$. See also Lemma 1.12 in *Pang* [18] for details.

First, it's clear that constant 0 is a subsolution of the above equation. Moreover, under assumption (4.16), it's easy to show that for each fixed $t\geq 0$, the equation:

$$\frac{2\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2}{\sigma_S^2}x^2-2\lambda x+k=0$$

has two positive real roots

$$x_1=\frac{\lambda-\sqrt{\lambda^2-\frac{2(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}k}}{2\frac{(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}},\quad\text{and}\quad x_2=\frac{\lambda+\sqrt{\lambda^2-\frac{2(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}k}}{2\frac{(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}}.$$

And for any positive constant a such that:

$$0<a<\frac{\lambda+\sqrt{\lambda^2-\frac{(\Theta_2+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}k}}{2\frac{(\Theta_1+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}},$$

with $\Theta_1=\max(\theta,\theta^*)$ and $\Theta_2=\min(\theta,\theta^*)$, and the positive constant b such that:

$$b>a\frac{(\Theta_1+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}-\frac{\lambda^2\bar{\mu}^2a^2}{2a^2\frac{(\Theta_1+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}-2a\lambda+k}.$$

It's easy to verify that $f(t,\eta)=\exp(bt+a\eta^2)$ satisfies:

$$f_t\geq\frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}f_{\eta\eta}-\lambda(\eta-\bar{\mu})f_\eta+k\eta^2,$$

with the initial condition $f(0,\eta)\geq 1$.

And since $Q_n(\eta)<k\eta^2$, we get function $f(t,\eta)$ is the supersolution of the equation (4.17), and $\big\langle 0,f(t,\eta)\big\rangle$ is the coupled subsolution and supersolution. Theorem 7.2 from *Pao* [19] shows that function $\phi(t,\eta)$ satisfies: $0\leq\phi(t,\eta)\leq f(t,\eta)\equiv\Lambda_1$, and hence Monotone Convergence Theorem leads to:

$$\mathbb{E}\Big[\exp\Big(\int_0^t k\hat{\mu}_s^2ds\Big)\Big]\leq\Lambda_1<+\infty.$$

□

Lemma 4.2. *If constant $\bar{k}>0$ satisfies*

$$\bar{k}<\frac{\lambda\sigma_S^2}{(\Theta+\sigma_S\sigma_\mu\rho)^2},\tag{4.18}$$

for fixed constant $\kappa>0$, there exists a constant Λ_2 independent of t, and

$$\mathbb{E}\Big[\exp\Big(\bar{k}(\hat{\mu}_t+\kappa)^2\Big)\Big]\leq\Lambda_2<\infty,\quad t\in[0,T].$$

Proof. Similar to the proof of Lemma 4.1, we again construct an increasing sequence of functions $\{Q_n(y)\}$ for $n\in\mathbb{N}$ such that $\lim_{n\to+\infty}Q_n(y)=\bar{k}(y+\kappa)^2$. And for each fixed $t\in[0,T]$ and η, we define:

$$\psi(t,\eta)=\mathbb{E}\Big[\exp\Big(Q_n(\hat{\mu}_t)\Big)\Big],$$

where $\hat{\mu}_0 = \eta$.

Then a direct corollary of Theorem 5.6.1 of *Friedman* [9] gives the function $\psi(t,\eta)$ is a classical solution of the linear parabolic equation:

$$\psi_t = \frac{\left(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\right)^2}{2\sigma_S^2}\psi_{\eta\eta} - \lambda(\eta - \bar{\mu})\psi_\eta, \tag{4.19}$$

with initial condition $\psi(0,\eta) = e^{Q_n(\eta)}$.

Under assumption (4.18), and choose any constant a such that

$$\bar{k} < a < \frac{\lambda\sigma_S^2}{(\Theta + \sigma_S\sigma_\mu\rho)^2},$$

where $x_1 = \frac{\lambda\sigma_S^2}{(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}$ is one real root of the algebraic equation:

$$\frac{2\left(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\right)^2}{\sigma_S^2}x^2 - 2\lambda x = 0$$

for each fixed $t \in [0,T]$. And choose any positive constant b such that

$$\begin{aligned} b >& a\frac{(\Theta + \sigma_S\sigma_\mu\rho)^2}{\sigma_S^2} + \frac{2a^2(\Theta + \sigma_S\sigma_\mu\rho)^2\kappa^2}{\sigma_S^2} + 2a\lambda\bar{\mu}\kappa \\ &- \frac{\left(\frac{2a^2\kappa(\Theta+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2} - a\lambda\kappa - a\lambda\bar{\mu}\right)^2}{2a^2\frac{(\Theta+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2} - 2a\lambda}, \end{aligned}$$

It is easy to verify that $f(t,\eta) = \exp(bt + a(\eta+\kappa)^2)$ satisfies

$$f_t \geq \frac{\left(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\right)^2}{2\sigma_S^2}f_{\eta\eta} - \lambda(\eta - \bar{\mu})f_\eta,$$

with the initial condition $f(0,\eta) = e^{a(\eta+\kappa)^2} \geq \psi(0,\eta)$, hence we get the function $f(t,\eta)$ is the supersolution of the equation (4.19), and it is trivial to show $g(t,\eta) \equiv 0$ is the subsolution, therefore $\langle 0, f(t,\eta)\rangle$ are the coupled subsolution and supersolution. Again by Theorem 7.2 from *Pao* [19], that function $\psi(t,\eta)$ satisfies: $0 \leq \psi(t,\eta) \leq f(t,\eta) \leq e^{bT+a(\eta+\kappa)^2} \equiv \Lambda_2$, hence Monotone Convergence Theorem implies:

$$\mathbb{E}\Big[\exp\Big(\bar{k}(\hat{\mu}_t + \kappa)^2\Big)\Big] \leq \Lambda_2 < +\infty, \quad \forall t \in [0,T].$$

□

PROOF OF THEOREM 3.2, *CONTINUED.*

For any pair of admissible control $(\pi_t, c_t) \in \mathcal{A}$, similar to the case for $p < 0$, choose the same localizing sequence τ_n such that

$$V(0,x_0,z_0,\eta_0) \geq \mathbb{E}\Big[\int_0^{\tau_n\wedge T}\frac{(c_s - Z_s)^p}{p}ds\Big] + \mathbb{E}\Big[V(\tau_n\wedge T, X_{\tau_n\wedge T}, Z_{\tau_n\wedge T}, \hat{\mu}_{\tau_n\wedge T})\Big]. \tag{4.20}$$

Now, by monotone convergence theorem, we first know:

$$\lim_{n\to+\infty}\mathbb{E}\Big[\int_0^{\tau_n\wedge T}\frac{(c_s-Z_s)^p}{p}ds\Big]=\mathbb{E}\Big[\int_0^{T}\frac{(c_s-Z_s)^p}{p}ds\Big].$$

And for $0<p<1$, $V(t,x,z,\eta)\geq 0$ for all $t\in[0,T]$ by the definition (3.32) and (4.1), and Fatou's lemma yields that:

$$\lim_{n\to+\infty}\mathbb{E}\Big[V(\tau_n\wedge T,X_{\tau_n\wedge T},Z_{\tau_n\wedge T},\hat{\mu}_{\tau_n\wedge T})\Big]\geq\mathbb{E}\Big[V(T,X_T,Z_T,\hat{\mu}_T)\Big]=\mathbb{E}\Big[\frac{X_T^p}{p}\Big],$$

which implies that:

$$V(0,x_0,z_0,\eta_0)\geq\sup_{\pi,c\in\mathcal{A}}\mathbb{E}\Big[\int_0^T\frac{(c_s-Z_s)^p}{p}ds+\frac{X_T^p}{p}\Big]=v(x_0,z_0,\eta_0,\theta_0).$$

On the other hand, for the π_t^* and c_t^* defined by (3.36), (3.37), again follow the same procedure in the proof for case $p<0$, we can show π_t^* and c_t^* are actually in the admissible space $\mathcal{A}$.

Now, by policies π_t^* and c_t^*, similarly, we can now get the equality:

$$V(0,x_0,z_0,\eta_0)=\mathbb{E}\Big[\int_0^{\tau_n\wedge T}\frac{(c_s^*-Z_s^*)^p}{p}ds\Big]+\mathbb{E}\Big[V(\tau_n\wedge T,X^*_{\tau_n\wedge T},Z^*_{\tau_n\wedge T},\hat{\mu}_{\tau_n\wedge T})\Big].$$

By the definition of $V(t,x,z,\eta)$, we know that:

$$V(T\wedge\tau_n,X^*_{T\wedge\tau_n},Z^*_{T\wedge\tau_n},\hat{\mu}_{T\wedge\tau_n})\leq k_1\Big[\Big(\frac{Y^*}{N}\Big)^{2p}_{T\wedge\tau_n}+N^2(T\wedge\tau_n,\hat{\mu}_{T\wedge\tau_n})\Big]$$

for some positive constants k_1, which are independent of n.

For the first term, we notice that

$$\begin{aligned}\Big(\frac{Y^*}{N}\Big)^{2p}_{T\wedge\tau_n}\leq&\frac{1}{2}\Big[\exp\Big(\int_0^{T\wedge\tau_n}\frac{4p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{W}_u-\int_0^{T\wedge\tau_n}\frac{4p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\\&+\exp\Big(\int_0^{T\wedge\tau_n}\frac{2(p^2+p)\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big]\end{aligned}$$

and hence, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_n\Big(\frac{Y^*}{N}\Big)^{2p}_{T\wedge\tau_n}\Big]\leq&\frac{1}{2}\mathbb{E}\Big[\sup_n\exp\Big(\int_0^{T\wedge\tau_n}\frac{4p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{W}_u-\int_0^{T\wedge\tau_n}\frac{4p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\\&+\sup_n\exp\Big(\int_0^{T\wedge\tau_n}\frac{2(p^2+p)\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big]\end{aligned}$$

and again since $\hat{\mu}_t$ follows the dynamics (3.2), by Beneš' condition (see Corollary 3.5.14 and Corollary 3.5.16 in *Karatzas and Shreve* [12]), we see that the exponential local martingale $M_t=\exp\Big(\int_0^t\frac{2p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{W}_u-\int_0^t\frac{2p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)$ is a true martingale, and hence, by Doob's maximal inequality,

we first derive that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_n \exp\Big(\int_0^{T\wedge\tau_n} \frac{4p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{W}_u - \int_0^{T\wedge\tau_n} \frac{4p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big]\\
\leq&\mathbb{E}\Big[\sup_{t\in[0,T]} \exp\Big(\int_0^{t} \frac{4p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{W}_u - \int_0^{t} \frac{4p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big]\\
\leq&k(p)\mathbb{E}\Big[\exp\Big(\int_0^{T} \frac{4p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{W}_u - \int_0^{T} \frac{4p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big]
\end{aligned}$$

where $k(p)$ is a constant depending on p. Moreover, similar to the proofs of Corollary 3.5.14 and Corollary 3.5.16 in *Karatzas and Shreve* [12], Corollary 1 and Corollary 2 in *Grigelionis and Mackevičius* [10] further states that the true martingale M_t defined as above satisfies the finite moments property, i.e., for any $r>1$, we have $\mathbb{E}\Big[M_T^r\Big]<\infty$. Hence we can conclude that for $r=2$,

$$\mathbb{E}\Big[\exp\Big(\int_0^{T} \frac{4p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{W}_u - \int_0^{T} \frac{4p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big]<\infty.$$

For the second part, we can therefore apply Assumption (3.33) and Lemma 4.1, and it yields that:

$$\begin{aligned}
\mathbb{E}\Big[\sup_n \exp\Big(\int_0^{T\wedge\tau_n} \frac{(2p^2+2p)\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big] \leq \mathbb{E}\Big[\exp\Big(\int_0^{T} \frac{(2p^2+2p)\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big]&\\
<\Lambda_1<+\infty,&
\end{aligned}$$

for some constant $\Lambda_1>0$.

We now recall that under the assumption that $a(t;s)$, $b(t;s)$, $c(t;s)$, $f(t;s)$ and $g(t;s)$ defined in Theorem 3.1 are bounded and $1-a(t;s)\hat{\Omega}(t)\neq 0$ on $0\leq t\leq s\leq T$, functions $A(t;s)$, $B(t;s)$ and $C(t;s)$ are bounded on $0\leq t\leq s\leq T$ and therefore there exists constants k, k_1 such that

$$N(t,\eta)\leq ke^{\bar{K}_1(\eta+k_1)^2},$$

where $A(t;s)\leq\bar{K}_1$ for all $0\leq t\leq s\leq T$, and hence we have

$$\sup_n\Big(N^2(T\wedge\tau_n,\hat{\mu}_{T\wedge\tau_n})\Big)\leq\sup_{t\in[0,T]}ke^{2\bar{K}_1(\hat{\mu}_t+k_1)^2}.$$

Then we just need to show that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}ke^{2\bar{K}_1(\hat{\mu}_t+k_1)^2}\Big]<\infty. \tag{4.21}$$

Define $\varphi(x)\triangleq e^{2\bar{K}_1(x+k_1)^2}$ and apply Itô's lemma, we have

$$\begin{aligned}
d\varphi(\hat{\mu}_t)=&\varphi(\hat{\mu}_t)\Big[\Big(-4\bar{K}_1\lambda+8\bar{K}_1^2\frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2}{\sigma_S^2}\Big)\hat{\mu}_t^2+4\bar{K}_1\lambda\bar{\mu}\hat{\mu}_t\\
&+2\bar{K}_1\frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2}{\sigma_S^2}\Big]dt+dL_t.
\end{aligned}$$

Assumption (3.34) guarantees $-4\bar{K}_1\lambda + 8\bar{K}_1^2 \frac{\left(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\right)^2}{\sigma_S^2} < 0$, and hence there exists an upper bound constant $k_2 > 0$ such that

$$d\varphi(\hat{\mu}_t) \leq \varphi(\hat{\mu}_t)k_2 dt + dL_t,$$

where the local martingale part is:

$$dL_t \triangleq \varphi(\hat{\mu}_t)4\bar{K}_1\hat{\mu}_t \frac{\left(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\right)}{\sigma_S} d\hat{W}_t.$$

From which we can derive that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \varphi(\hat{\mu}_t)\Big] \leq \varphi(\eta) + \int_0^T k_2 \mathbb{E}\Big[\sup_{s\in[0,t]} \varphi(\hat{\mu}_s)\Big] dt + \mathbb{E}\Big[\sup_{t\in[0,T]} L_t\Big],$$

Burhholder-Davis-Gundy Inequality and Jensen's Inequality induce that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} L_t\Big] \leq k_3 \Big(\int_0^T \mathbb{E}\Big[16\bar{K}_1^2 \frac{\left(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\right)^2}{\sigma_S^2} \hat{\mu}_t^2 e^{4\bar{K}_1(\hat{\mu}_t+k_1)^2}\Big] dt\Big)^{\frac{1}{2}},$$

However, under Assumption (3.34), there exists a constant $\epsilon > 0$ such that $4\bar{K}_1 + \epsilon < \frac{\lambda\sigma_S^2}{(\Theta+\sigma_S\sigma_\mu\rho)^2}$, and by Hölder's Inequality, choose the conjugates $q = \frac{4\bar{K}_1+\epsilon}{4\bar{K}_1}$ and $\frac{1}{q} + \frac{1}{q'} = 1$, then

$$\mathbb{E}\Big[\hat{\mu}_t^2 e^{4\bar{K}_1(\hat{\mu}_t+k_1)^2}\Big] \leq \left(\mathbb{E}\Big[\hat{\mu}_t^{2q'}\Big]\right)^{\frac{1}{q'}} \left(\mathbb{E}\Big[e^{(4\bar{K}_1+\epsilon)(\hat{\mu}_t+k_1)^2}\Big]\right)^{\frac{1}{q}},$$

and by Lemma 4.2, there exists a constant Λ_2 independent of t such that

$$\mathbb{E}\Big[e^{(4\bar{K}_1+\epsilon)(\hat{\mu}_t+k_1)^2}\Big] \leq \Lambda_2 < \infty, \ \forall t \in [0,T].$$

And again by the fact that there exists positive constants l and $l_1 > 1$ large enough, such that:

$$\sup_{t\in[0,T]} \hat{\mu}_t \leq l + \sup_{t\in[0,T]} l_1\hat{W}_t, \quad t \in [0,T],$$

we also obtain

$$\int_0^T \left(\mathbb{E}\Big[\hat{\mu}_t^{2q'}\Big]\right)^{\frac{1}{q'}} dt \leq T\Big(\mathbb{E}\Big[(l + \sup_{t\in[0,T]} l_1\hat{W}_t)^{2q'}\Big]\Big)^{\frac{1}{q'}} < \infty,$$

due to the distribution of running maximum of the Brownian motion $\hat{W}_t$. Hence we get the boundedness of $\mathbb{E}\Big[\sup_{t\in[0,T]} L_t\Big] \leq k_4 < \infty$ for some constant k_4, and

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \varphi(\hat{\mu}_t)\Big] \leq \varphi(\eta) + \int_0^T k_2 \mathbb{E}\Big[\sup_{s\in[0,t]} \varphi(\hat{\mu}_s)\Big] dt + k_4,$$

The Gronwall's Inequality verifies (4.21).

Therefore, putting all pieces together, we eventually derived that

$$\mathbb{E}\Big[\sup_n V(\tau_n \wedge T, X_{\tau_n\wedge T}, Z_{\tau_n\wedge T}, \hat{\mu}_{\tau_n\wedge T})\Big] < \infty \tag{4.22}$$

and Dominated Convergence Theorem leads to:

$$\lim_{n\to\infty} \mathbb{E}\Big[V(\tau_n \wedge T, X_{\tau_n\wedge T}, Z_{\tau_n\wedge T}, \hat{\mu}_{\tau_n\wedge T})\Big] = \mathbb{E}\Big[\frac{(X_T^*)^p}{p}\Big].$$

Together with Monotone Convergence Theorem, we deduce

$$V(0, x_0, z_0, \eta_0; \theta_0) = \mathbb{E}\Big[\int_0^T \frac{(c_s^* - Z_s^*)^p}{p} ds + \frac{(X_T^*)^p}{p}\Big] \leq v(x_0, z_0, \eta_0, \theta_0),$$

which completes the proof. □

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Appendix A. Fully Explicit Solutions to the Auxiliary ODEs

Follow the arguments by *Kim and Omberg* [13], we can even solve the auxiliary ODEs (3.25), (3.26), (3.27), (3.28) and (3.29) fully explicitly depending on the risk aversion constant p and all the market coefficients σ_S, σ_μ, λ, ρ:

A.1. **The Normal Solution.**

The condition for the Normal solution is

$$\Delta \triangleq \lambda^2 - \frac{2\lambda p\rho\sigma_\mu}{(1-p)\sigma_S^2} - \frac{p\sigma_\mu^2}{(1-p)\sigma_S^2} > 0, \tag{A.1}$$

and then we define:

$$\xi = \sqrt{\Delta} = \sqrt{\gamma_2^2 - \gamma_1\gamma_3}, \qquad \gamma_1 = \frac{(1-p+p\rho^2)}{1-p}\sigma_\mu^2,$$

$$\gamma_2 = -\lambda + \frac{p\rho\sigma_\mu}{(1-p)\sigma_S}, \qquad \gamma_3 = \frac{p}{(1-p)\sigma_S^2},$$

$$\xi_1 = \frac{\sqrt{(1-\rho^2)\sigma_\mu^2 + (\lambda\sigma_S + \rho\sigma_\mu)^2}}{\sigma_S}.$$

We can solve the equations (3.25), (3.26), (3.27), (3.28) and (3.29) as:

$$a(t;s) = \frac{p(1-e^{2\xi(t-s)})}{2(1-p)\sigma_S^2\Big[2\xi - (\xi+\gamma_2)(1-e^{2\xi(t-s)})\Big]},$$

$$b(t;s) = \frac{p\lambda\bar{\mu}(1-e^{\xi(t-s)})^2}{2(1-p)\sigma_S^2\xi\Big[2\xi - (\xi+\gamma_2)(1-e^{2\xi(t-s)})\Big]},$$

$$\begin{aligned} c(t;s) =& \frac{p}{2(1-p)\sigma_S^2}\Big(\frac{\lambda^2\bar{\mu}^2}{\xi^2} - \frac{\sigma_\mu^2}{\xi+\gamma_2}\Big)(s-t) + \frac{p\lambda^2\bar{\mu}^2\Big[(\xi+2\gamma_2)e^{2\xi(t-s)} - 4\gamma_2 e^{\xi(t-s)} + 2\gamma_2 - \xi\Big]}{2(1-p)\sigma_S^2\xi^3\Big[2\xi - (\xi+\gamma_2)\Big(1-e^{2\xi(t-s)}\Big)\Big]} \\ & - \frac{p\sigma_\mu^2}{2(1-p)\sigma_S^2(\xi^2-\gamma_2^2)}\log\Big|\frac{2\xi - (\xi+\gamma_2)(1-e^{2\xi(t-s)})}{2\xi}\Big|, \end{aligned}$$

$$
\begin{aligned}
f(t;s) =& -\frac{1}{2\sigma_S}\frac{1-e^{2\xi_1(t-s)}}{(\sigma_S\xi_1+\lambda\sigma_S+\rho\sigma_\mu)+(\sigma_S\xi_1-\lambda\sigma_S-\rho\sigma_\mu)e^{2\xi_1(t-s)}},\\
g(t;s) =& \frac{1}{2}\log\Big(\frac{(\sigma_S\xi_1+\lambda\sigma_S+\rho\sigma_\mu)+(\sigma_S\xi_1-\lambda\sigma_S-\rho\sigma_\mu)e^{2\xi_1(t-s)}}{2\sigma_S\xi_1 e^{\xi_1(t-s)}}\Big)\\
&-\frac{(1-p)(1-\rho^2)}{2(1-p+p\rho^2)}\log\Big(\frac{(\sigma_S\xi+\lambda\sigma_S+\frac{\rho\sigma_\mu p}{1-p})+(\sigma_S\xi-\lambda\sigma_S-\frac{\rho\sigma_\mu p}{1-p})e^{2\xi(t-s)}}{2\sigma_S\xi e^{\xi(t-s)}}\Big)\\
&-\frac{\rho^2\lambda(s-t)}{2(1-p+p\rho^2)}-\frac{\rho\sigma_\mu(s-t)}{2(1-p+p\rho^2)\sigma_S},
\end{aligned}
$$

The condition for the bounded Normal solution is

$$\gamma_3>0,\quad \text{or}\quad \gamma_1>0,\quad \text{or}\quad \gamma_2<0. \tag{A.2}$$

The condition for the explosive solution and the critical point is

$$
\begin{gathered}
\gamma_3<0,\quad \gamma_1<0,\quad \text{and}\quad \gamma_2>0,\\
s-t=\frac{1}{2\xi}\log\Big(\frac{\gamma_2+\xi}{\gamma_2-\xi}\Big).
\end{gathered}
$$

Remark A.1. *By the observation, if $p<0$, the conditions* (A.1) *and* (A.2) *hold, and we have $a(t;s)\le 0$ is a bounded solution as well as $1-2a(t;s)\hat{\Omega}(t)>1>0$ and $1-f(t;s)\hat{\Omega}(t)>1>0$, hence we can finally conclude the solutions of ODEs* (3.22), (3.23), (3.24) *are all bounded on $0\le t\le s\le T$. We also notice that $A(t)=\frac{a(t)}{(1-p)(1-2a(t)\hat{\Omega}(t))}\le 0$, on $0\le t\le s\le T$.*

A.2. **The Hyperbolic Solution.** The condition for the Hyperbolic solution is

$$\Delta \triangleq \lambda^2-\frac{2\lambda p\rho\sigma_\mu}{(1-p)\sigma_S^2}-\frac{p\sigma_\mu^2}{(1-p)\sigma_S^2}=0,$$

together with

$$\gamma_2=-\lambda+\frac{p\rho\sigma_\mu}{(1-p)\sigma_S}\neq 0,$$

Then we can solve (3.25), (3.26), (3.27), (3.28) and (3.29) as:

$$
\begin{aligned}
a(t;s) =& \frac{-1}{2\gamma_1(s-t-\frac{1}{\gamma_2})}-\frac{\gamma_2}{2\gamma_1},\\
b(t;s) =& -\frac{2\lambda\bar{\mu}}{4\gamma_1\gamma_2(s-t-\frac{1}{\gamma_2})}-\frac{\gamma_2\lambda\bar{\mu}(s-t+\frac{1}{\gamma_2})}{2\gamma_1},\\
c(t;s) =& \frac{\gamma_2\sigma_\mu^2(s-t)}{2\gamma_1}+\frac{\lambda^2\bar{\mu}^2\gamma_2^2(s-t-\frac{4}{\gamma_2})(s-t)^3}{24\gamma_1(s-t-\frac{1}{\gamma_2})}+\frac{\sigma_\mu^2\log\Big|\frac{1}{2}(s-t)\gamma_2-1\Big|}{\gamma_1},
\end{aligned}
$$

$$
\begin{aligned}
f(t;s) =& -\frac{1}{2\sigma_S}\frac{1-e^{2\xi_1(t-s)}}{(\sigma_S\xi_1+\lambda\sigma_S+\rho\sigma_\mu)+(\sigma_S\xi_1-\lambda\sigma_S-\rho\sigma_\mu)e^{2\xi_1(t-s)}},\\
g(t;s) =& \frac{1}{2}\log\Big(\frac{(\sigma_S\xi_1+\lambda\sigma_S+\rho\sigma_\mu)+(\sigma_S\xi_1-\lambda\sigma_S-\rho\sigma_\mu)e^{2\xi_1(t-s)}}{2\sigma_S\xi_1 e^{\xi_1(t-s)}}\Big)-\frac{(\lambda\sigma_S+\rho\sigma_\mu)}{2\sigma_S}(s-t)\\
&+\frac{\sigma_\mu^2(1-\rho^2)}{2\gamma_1}\Big[\log\Big|1+\gamma_2(t-s)\Big|-\gamma_2(s-t)\Big].
\end{aligned}
$$

The condition for the bounded Hyperbolic solution is

$$\gamma_2 < 0.$$

The condition for the explosive solution and the critical point is

$$\gamma_2 > 0, \quad \text{and} \quad s - t = \frac{1}{\gamma_2}.$$

A.3. **The Polynomial solution.** The condition for the Polynomial solution is

$$\Delta \triangleq \lambda^2 - \frac{2\lambda p\rho\sigma_\mu}{(1-p)\sigma_S^2} - \frac{p\sigma_\mu^2}{(1-p)\sigma_S^2} = 0,$$

together with

$$\gamma_2 = -\lambda + \frac{p\rho\sigma_\mu}{(1-p)\sigma_S} = 0,$$

Then we can solve (3.25), (3.26), (3.27), (3.28) and (3.29) as:

$$\begin{aligned}
a(t;s) =& \frac{p}{2(1-p)\sigma_S^2}(s-t), \\
b(t;s) =& \frac{p}{2(1-p)\sigma_S^2}\lambda\bar{\mu}(s-t)^2, \\
c(t;s) =& -\frac{p}{4(1-p)\sigma_S^2}\sigma_\mu^2(s-t)^2 + \frac{p}{6(1-p)\sigma_S^2}\lambda^2\bar{\mu}^2(s-t)^3,
\end{aligned}$$

$$\begin{aligned}
f(t;s) =& -\frac{1}{2\sigma_S}\frac{1-e^{2\xi_1(t-s)}}{(\sigma_S\xi_1 + \lambda\sigma_S + \rho\sigma_\mu) + (\sigma_S\xi_1 - \lambda\sigma_S - \rho\sigma_\mu)e^{2\xi_1(t-s)}}, \\
g(t;s) =& \frac{1}{2}\log\Big(\frac{(\sigma_S\xi_1 + \lambda\sigma_S + \rho\sigma_\mu) + (\sigma_S\xi_1 - \lambda\sigma_S - \rho\sigma_\mu)e^{2\xi_1(t-s)}}{2\sigma_S\xi_1 e^{\xi_1(t-s)}}\Big) - \frac{(\lambda\sigma_S + \rho\sigma_\mu)}{2\sigma_S}(s-t) \\
& - \frac{\sigma_\mu^2(1-\rho^2)p}{4(1-p)\sigma_S^2}(s-t)^2.
\end{aligned}$$

All Polynomial solutions are bounded.

A.4. **The Tangent solution.** The condition for the Tangent solution is

$$\Delta \triangleq \lambda^2 - \frac{2\lambda p\rho\sigma_\mu}{(1-p)\sigma_S^2} - \frac{p\sigma_\mu^2}{(1-p)\sigma_S^2} < 0,$$

Now, we define

$$\zeta = \sqrt{-\Delta}, \quad \varpi = \tan^{-1}\Big(\frac{\gamma_2}{\zeta}\Big),$$

Then we can solve (3.25), (3.26), (3.27), (3.28) and (3.29) as:

$$\begin{aligned}
a(t;s) =& \frac{\zeta}{2\gamma_1}\tan\Big(\zeta(s-t) + \varpi\Big) - \frac{\gamma_2}{2\gamma_1}, \\
b(t;s) =& \frac{\lambda\bar{\mu}}{\gamma_1}\Big[-1 - \tan(\varpi)\tan(\zeta(s-t)+\varpi) + \sec(\varpi)\sec(\zeta(s-t)+\varpi)\Big],
\end{aligned}$$

$$
\begin{aligned}
c(t;s) =& \frac{2\lambda^2\bar{\mu}^2\gamma_2\sqrt{\gamma_2^2+\zeta^2}}{2\gamma_1\zeta}\Big[\sec(\varpi) - \sec(\zeta(s-t)+\varpi)\Big] \\
&+ \frac{\lambda^2\bar{\mu}^2(2\gamma_2+\zeta^2)}{2\gamma_1\zeta^3}\Big[\tan(\zeta(s-t)+\varpi) - \tan(\varpi)\Big] \\
&- \frac{\lambda^2\bar{\mu}^2(\gamma_2^2+\zeta^2) - \gamma_2\zeta^2\sigma_\mu^2}{2\gamma_1\zeta^2} + \frac{\sigma_\mu^2}{2\gamma_1}\log\Big(\sec(\varpi)\cos(\zeta(s-t)+\varpi)\Big), \\
f(t;s) =& -\frac{1}{2\sigma_S}\frac{1-e^{2\xi_1(t-s)}}{(\sigma_S\xi_1+\lambda\sigma_S+\rho\sigma_\mu)+(\sigma_S\xi_1-\lambda\sigma_S-\rho\sigma_\mu)e^{2\xi_1(t-s)}}, \\
g(t;s) =& \frac{1}{2}\log\Big(\frac{(\sigma_S\xi_1+\lambda\sigma_S+\rho\sigma_\mu)+(\sigma_S\xi_1-\lambda\sigma_S-\rho\sigma_\mu)e^{2\xi_1(t-s)}}{2\sigma_S\xi_1e^{\xi_1(t-s)}}\Big) - \frac{(\lambda\sigma_S+\rho\sigma_\mu)}{2\sigma_S}(s-t) \\
&- \sigma_\mu^2(1-\rho^2)\Big[\frac{1}{2\gamma_1}\log\Big|\frac{\cos(\zeta(t-s)+\varpi)}{\cos(\varpi)}\Big| - \frac{\gamma_2}{2\gamma_1}(s-t)\Big].
\end{aligned}
$$

All Tangent solutions are explosive solutions and the critical point is

$$
s-t = \frac{\pi}{2\zeta} - \frac{1}{\zeta}\tan^{-1}(\frac{\gamma_2}{\zeta}).
$$

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